Math 250: 4.5 Linear Approximation and Differentials

Objectives:

1) Understand that differentiability also means local linearity.
   a. If $f'(a)$ exists, we can find a tangent line to f at $x = a$, $y - f(a) = f'(a) \cdot (x - a)$
   b. If we graph f and the tangent line together, and zoom in near $x = a$, the function and the tangent line are very close together.
2) Estimate the function value using the tangent line (linear approximation).
   a. When we isolate y
      i. $y - f(a) = f'(a) \cdot (x - a)$ becomes
      ii. $y = f(a) + f'(a) \cdot (x - a)$
   b. Call this y the linear approximation $L(x) = f(a) + f'(a) \cdot (x - a)$
   c. For values of x sufficiently close to a, $f(x) \approx L(x)$
   d. If x gets too far from a, the approximation is not very accurate.
3) Use the second derivative $f''(a)$ to describe the accuracy of $f(x) \approx L(x)$
   a. The sign of $f''(a)$ tells if the estimate is too high (overestimate) or too low (underestimate)
      i. If $f''(a) > 0$, f is concave up, curving upward from the tangent line. The linear approximation is beneath the function, and underestimates the value of the function.
      ii. If $f''(a) < 0$, f is concave down, curving downward from the tangent line. The linear approximation is above the function, and overestimates the value of the function.
   b. The size of the curvature $|f''(a)|$ tells if the estimate is valid over a large or small interval
      i. If $|f''(a)|$ is large, the function curves away from the tangent line quickly, making the approximation $f(x) \approx L(x)$ valid for only a small interval of x-values near a.
      ii. If $|f''(a)|$ is small, the function remains close to the tangent line, making the approximation $f(x) \approx L(x)$ valid for a larger interval of x-values near a.
4) Estimate the *change* in the function using the *change* in the linear approximation.
   a. As changes from a to x, want the change in the function: $\Delta y = \Delta f = f(x) - f(a)$
   b. Change in x is called $\Delta x = x - a$
   c. Approximate $\Delta y \approx L(x) - L(a)$
   d. But $\Delta y \approx L(x) - L(a)$ simplifies a lot!

$$\begin{aligned}
\Delta y &\approx L(x) - L(a) \\
&= [f(a) + f'(a)(x-a)] - [f(a) + f'(a)(a-a)] \\
&= f(a) + f'(a)(x-a) - f(a) + f'(a) \cdot 0 \\
&= f'(a)(x-a)
\end{aligned}$$

So:

$$\begin{aligned}
\Delta y &\approx f'(a)(x-a) \\
\Delta y &\approx f'(a) \cdot \Delta x
\end{aligned}$$

5) Use differential notation to estimate Δy by the change in the linear approximation.
   a. Define $dy = f'(a) \cdot dx$ and $dx = \Delta x$, so $dy = f'(a) \cdot dx$
   b. The approximation becomes $\Delta y \approx dy$
   c. This works for variable values of a, giving $dy = f'(x) \cdot dx$
      i. CAUTION: Most of the time we mean a specific value of $x = a$ and $f'(a)$

Examples and Practice:

1) Use $f(x) = \sqrt[3]{x}$
   a. Find the linear approximation to $f(x) = \sqrt[3]{x}$ for values of x near $a = 1$
   b. Use the linear approximation to estimate $\sqrt[3]{2}$
   c. Compare the approximation to the exact value of $\sqrt[3]{2}$ and calculate the error.
   d. Find the linear approximation to $f(x) = \sqrt[3]{x}$ for values of x near $a = 27$
   e. Use the linear approximation to estimate $\sqrt[3]{26}$
   f. Compare the approximation to the exact value of $\sqrt[3]{26}$ and calculate the error.
   g. Use f'' to identify if these approximations are overestimates or underestimates.
   h. Use the curvature to explain the accuracy of these two estimates.
2) Use $y = f(x) = x^9 - 2x + 1$
   a. Approximate the change in $y = f(x) = x^9 - 2x + 1$ when x changes from 1.00 to 1.05.
   b. Express this change using differential notation.
3) Approximate the change in the surface are of a spherical balloon when the radius decreases from 4 m to 3.9 m. Express this change using differential notation.
4) Find the differential form of the derivative of $f(x) = 3\cos^2(x)$

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① $f(x) = \sqrt[3]{x}$

a) Linear approximation near $a = 1$.

$$L(x) = f(a) + f'(a)(x - a)$$

$a = 1$

Find component parts of $L(x)$.

$f(a) = \sqrt[3]{1} = 1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

$f'(1) = \frac{1}{3\sqrt[3]{1^2}} = \frac{1}{3 \cdot 1} = \frac{1}{3}$

Plug in:

$L_1(x) = 1 + \frac{1}{3}(x - 1)$

$= 1 + \frac{1}{3}x - \frac{1}{3}$ simplify

$\boxed{L_1(x) = \frac{1}{3}x + \frac{2}{3}}$

b) Estimate $\sqrt[3]{2}$ using $L_1(x)$ found in a).

$f(2) = \sqrt[3]{2}$ so $x = 2$

$L_1(2) = \frac{1}{3}(2) + \frac{2}{3} = \frac{4}{3} = \boxed{1.\overline{3}}$

c) Compare to exact $\sqrt[3]{2}$ and calculate error.

GC $\boxed{\sqrt[3]{2} \approx 1.25992105}$

error $= \left| L_1(2) - \sqrt[3]{2} \right|$

$= \left| \frac{4}{3} - \sqrt[3]{2} \right|$

$\approx \boxed{.0734122834}$

d) Linear approximation near $a = 27$

$f(a) = f(27) = \sqrt[3]{27} = 3$

$f'(a) = f'(27) = \frac{1}{3(\sqrt[3]{27})^2} = \frac{1}{3 \cdot 3^2} = \frac{1}{27} = .037$

$L_2(x) = f(a) + f'(a)(x - a)$

$L_2(x) = 3 + \frac{1}{27}(x - 27)$

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$L_2(x) = 3 + \frac{1}{27}x - 1$

$\boxed{L_2(x) = \frac{1}{27}x + 2}$

e) Estimate $\sqrt[3]{26}$ using $L_2(x)$ found in d).

$L_2(26) = \frac{1}{27}(26) + 2$

$= \frac{26}{27} + 2$

$= \boxed{\frac{80}{27} = 2.\overline{962}}$

f) Compare to exact $\sqrt[3]{26}$ and calculate error.

GC $\sqrt[3]{26} \approx \boxed{2.962496068}$

$\text{error} = \left| L_2(26) - \sqrt[3]{26} \right|$

$= \left| \frac{80}{27} - \sqrt[3]{26} \right|$

$\approx \boxed{.000466894556}$

g) Use f'' to identify if each approximation is an overestimate or an underestimate.

$f''(x) = \frac{d}{dx}\left[\frac{1}{3}x^{-2/3} \right]$

$= \frac{1}{3}\left(\frac{-2}{3}\right)x^{-2/3-1}$

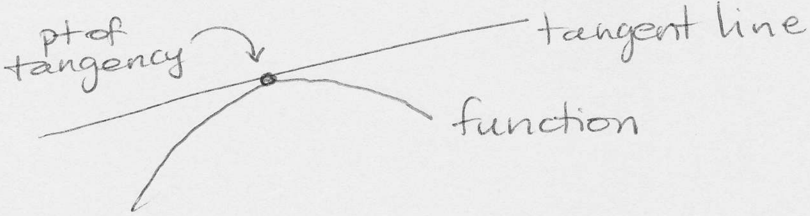
$= \frac{-2}{9}x^{-5/3}$

$= \frac{-2}{9(\sqrt[3]{x})^5}$

$f''(1) = \frac{-2}{9(\sqrt[3]{1})^5} = \frac{-2}{9} = -.\overline{2}$

$f''(27) = \frac{-2}{9(\sqrt[3]{27})^5} = \frac{-2}{2187} \approx -.000914494742$

Both $f''(1)$ and $f''(27)$ are negative, so f is concave down at both $a=1$ and $a=27$



So both approximations are $\boxed{\text{overestimates}}$, because the y-coordinates of the tangent line are higher (above) the y-coordinates of the function.

h) Use curvature to explain the accuracy of these estimates.

$|f''(1)| = .\overline{2}$

$|f''(27)| \approx .0009$

f has greater curvature at $a=1$ than it does at $a=27$, because $.2 > .0009$.
So even though $\sqrt[3]{2}$ is $x=2$, one unit from $a=1$
and $\sqrt[3]{26}$ is $x=26$, one unit from $a=27$,
the function curved away from the tangent line more quickly near $a=1$ than it did near $a=27$, making the approximate $L(2)$ have greater error than the approximation $L_2(26)$

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(2) Use $f(x) = x^9 - 2x + 1$.

a) Approximate the change in $y = f(x)$ when x changes from 1.00 to 1.05.

$\Delta y \approx f'(a) \cdot \Delta x$ approximating the change in the function using the change in the linear approximation.

$f'(x) = 9x^8 - 2$

$\Delta x = x - a$ $\quad a = 1.00$

$x = 1.05$

$= 1.05 - 1$

$\Delta x = .05$

$\Delta y \approx f'(1.00) \cdot (0.05)$

$= (9(1)^8 - 2)(0.05)$

$= 7(.05)$

$= \boxed{.35}$

b) Express in differential notation

$dy = f'(a) \cdot dx$

$\boxed{dy = .35}$

By the way, we did not calculate the actual change

$\Delta y = f(x) - f(a)$

$= f(1.05) - f(1)$

$= [(1.05)^9 - 2(1.05) + 1] - [1^9 - 2(1) + 1]$

$\approx \boxed{.451328216\ldots}$

exact: $\left[\left(\frac{21}{20}\right)^9 - 2\left(\frac{21}{20}\right) + 1\right] - 0$

$\left[\left(\frac{21}{20}\right)^9 - \frac{21}{10} + 1\right]$ = a brutal decimal.

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(3) Approximate the change in the surface area of a spherical balloon when the radius decreases from 4 m to 3.9 m.

$\Delta y \approx f'(a) \cdot \Delta x$

$f(x) =$ surface area of spherical balloon

$A(r) = 4\pi r^2$

so $A'(r) = 8\pi r$ plays role of $f'(x)$.

$a = 4$ m

$x = 3.9$ m $\leftarrow r$

$\Delta x = x - a \leftarrow r - a$

$= 3.9 - 4$

$= -0.1$

* if Δx is negative, x has decreased

$$\Delta A = A'(4) \cdot (3.9 - 4)$$
$$= 8\pi(4) \cdot (-0.1)$$
$$= \boxed{-3.2\pi \text{ m}^2} \quad \text{exact calculation}$$
$$\approx 10.0531 \text{ m}^2 \quad \text{approximate calculation}$$

Express using differential notation

$$dA = A'(r) \cdot dr$$
$$dA = 8\pi r \cdot dr$$
$$\boxed{dA = -3.2\pi \text{ m}^2}$$

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(4) Find the differential form of the derivative of $f(x) = 3\cos^2(x)$.

$$f'(x) = \frac{dy}{dx} = 3 \cdot 2 \cdot \underbrace{[\cos(x)]^1 \cdot (-\sin(x))}_{\text{chain rule}}$$

$$\frac{dy}{dx} = -6\cos(x)\sin(x)$$

$$\boxed{dy = -6\cos(x)\sin(x)\,dx}$$

Note: $\sin(2x) = 2\cos(x)\sin(x)$

so $-3\sin(2x) = -6\cos(x)\sin(x)$

making another (better) answer:

$$\boxed{dy = -3\sin(2x)\,dx}$$